

Fabio Siringo

*Dipartimento di Fisica e Astronomia, Università di Catania,*

*Corso Italia 57, I 95129 Catania, Italy*

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## Abstract

The mass and the decay width of a Higgs boson in the minimal standard model are evaluated by a variational method in the limit of strong self-coupling interaction. The non-perturbative technique provides an interpolation scheme between strong-coupling regime and weak-coupling limit where the standard perturbative results are recovered. In the strong-coupling limit the physical mass and the decay width of the Higgs boson are found to be very small as a consequence of mass renormalization. Thus it is argued that the eventual detection of a light Higgs boson would not rule out the existence of a strongly interacting Higgs sector.

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The impressive success of the Standard Model (SM) has enforced the common believing that the Higgs boson will be soon detected by the new generation of accelerators [1]. In fact there are two unknown parameters in the SM that wait for their experimental determination: the mass  $m$  of the Higgs boson and the strength of its self-coupling interaction  $\lambda$ . This last one determines the bare Higgs mass  $m_0^2 = \lambda v^2/3$  where  $v$  is the vacuum expectation value for the scalar field which is fixed by the known strength of weak interactions. Thus at tree level perturbation theory predicts a light Higgs mass  $m \approx m_0$  if the coupling  $\lambda$  is small enough. Conversely, in the strong coupling limit, perturbation theory breaks down and there is no simple relation between  $m$  and  $\lambda$ . A light weakly interacting Higgs boson has been strongly desired, mainly because perturbation theory would be reliable, and the Higgs boson would be detectable at a reasonable energy threshold. However, if nature had chosen for a strongly interacting boson, the physics would be richer and more interesting. Actually, the physics of such a strongly interacting Higgs boson has been explored in the last twenty years, and interesting proposals have been discussed ranging from the existence of bound states [2–8] to unconventional descriptions of the symmetry breaking mechanism [9].

During the last years the possibility of a strongly interacting Higgs boson has been rejected for two main reasons: i) A large  $\lambda$  is believed to imply a large mass, in contrast with the recent phenomenological evidence [1] for a light  $m \approx 100 - 200$  GeV; ii) For a strongly interacting Higgs boson the decay width  $\Gamma$  has been predicted to be very large [10,11] compared to the mass, and such very large resonance could hardly be regarded as a true particle. In this letter we point out that both the statements i) and ii) have a perturbative nature and cannot be trusted in the strong coupling limit. At tree level  $m$  and  $\Gamma$  are small if the coupling  $\lambda$  is small, which is consistent in the framework of perturbation theory. However if  $\lambda$  is very large any perturbative argument breaks down and fails to predict what  $m$  and  $\Gamma$  are. In fact, by use of a variational method we show that both  $m$  and  $\Gamma$  are small in the strong coupling limit.

The existence of a saturation of  $m$  at strong coupling has been shown by several non-perturbative techniques as  $1/N$  expansions [11], variational methods [12] and Bethe-Salpeter

equation [13]. We have shown that a further increase of the coupling strength yields a decrease of the mass [12], and this has also been confirmed by recent Bethe-Salpeter calculations [13]. The physical reason is very simple: at tree level  $m$  is proportional to  $\lambda$ ; however the interaction renormalizes the mass, since the attractive self-coupling reduces the energy of a free boson. At some stage this reduction overcomes the tree level increase, and the renormalized mass decreases for some very large self coupling. As a result a light Higgs boson could be a very strongly interacting particle whose ground state could even be a Higgs-Higgs bound state.

A light self-interacting Higgs boson would not make any sense as a free particle if its decay width  $\Gamma$  would be so large and increasing with  $\lambda$  as found by  $1/N$  expansion calculations [11]. However in the real world the goldstone bosons of the  $O(N)$  model do not play any physical role, while the Higgs sector is coupled with the gauge bosons through a quite weak interaction which does not increase with  $\lambda$ . As could be expected, we show that for very large couplings and a reasonable choice of the cut-off, a light Higgs boson would be characterized by a very small decay width: thus the experimental knowledge of  $m$  and  $\Gamma$  would not say the last word on the strength of the self-interaction. The eventual detection of a light Higgs with a narrow decay width would be consistent with both a perturbative weakly interacting and a non-perturbative strongly interacting theory.

In order to deal with the non-perturbative limit we use a variational method in the Hamiltonian formalism [14–16,6,8]. The method has the advantage of yielding the known perturbative results in the weak-coupling limit [16,8] (e.g. masses, decay widths and binding energies), while it can be safely extended to the non-perturbative strong coupling regime. The results achieved by such method have not been appreciated in the past since the variational equations have been usually approximated by perturbative methods [17] thus spoiling their most important advantages. In the framework of a study on bound states we have recently shown [12] that the variational equations can be decoupled exactly, giving important consequences on mass renormalization. In this letter we show that the same method can be used for decoupling the variational equations arising from a more complete trial state,

describing a Higgs field  $h$  which interacts with a neutral gauge vector field  $Z^\mu$ :

$$|\Psi\rangle = |h\rangle + |hh\rangle + |hhh\rangle + |ZZ\rangle \quad (1)$$

where

$$|h\rangle = A a_0^\dagger |0\rangle, \quad (2)$$

$$|hh\rangle = \int d^3p B(\mathbf{p}) a_{\mathbf{p}}^\dagger a_{-\mathbf{p}}^\dagger |0\rangle, \quad (3)$$

$$|hhh\rangle = \int d^3p d^3q d^3k G(\mathbf{p}, \mathbf{q}, \mathbf{k}) a_{\mathbf{p}}^\dagger a_{\mathbf{q}}^\dagger a_{\mathbf{k}}^\dagger |0\rangle \delta^3(\mathbf{p} + \mathbf{q} + \mathbf{k}), \quad (4)$$

$$|ZZ\rangle = \sum_{\sigma\sigma'} \int d^3p C_{\sigma\sigma'}(\mathbf{p}) b_{\mathbf{p}\sigma}^\dagger b_{-\mathbf{p}\sigma'}^\dagger |0\rangle. \quad (5)$$

Here  $a_{\mathbf{p}}^\dagger$  is the creation operator for a Higgs particle of momentum  $\mathbf{p}$  and mass  $m$ ,  $b_{\mathbf{p}\sigma}^\dagger$  is the creation operator for a neutral vector boson  $Z^0$  of momentum  $\mathbf{p}$ , polarization  $\sigma$  and mass  $M$ , and  $|0\rangle$  is the vacuum annihilated by the corresponding annihilation operators. The coefficients  $A, B, C, G$  can be determined from the variational principle

$$\delta\langle\Psi| : \hat{H} - E : |\Psi\rangle = 0. \quad (6)$$

All the required terms of the Hamiltonian  $\hat{H}$  can be canonically derived from the SM Lagrangian density

$$\begin{aligned} \mathcal{L} = & -\frac{1}{2}\partial_\mu h \partial^\mu h - \frac{1}{2}m_0^2 h^2 - \frac{1}{3!}\lambda v h^3 - \frac{1}{4!}\lambda h^4 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \\ & - \frac{1}{2}M^2 Z_\mu Z^\mu - \frac{M^2}{v}Z_\mu Z^\mu h - \frac{1}{2}\left(\frac{M}{v}\right)^2 Z_\mu Z^\mu h^2. \end{aligned} \quad (7)$$

This is the Lagrangian of a  $U(1)$  Higgs model (scalar electrodynamics) which is equivalent to the full SM Lagrangian as far as we only consider the trial state (1). The variational principle (6) yields four coupled integral equations (eigenvalue equations) for the coefficients  $A, B, C, G$ . The full equations have been reported in Ref. [8]. They can be considerably simplified by taking advantage of the symmetry properties of the bosons: without any

loss of generality the functions  $B$  and  $G$  may be taken to be even under spatial inversion, the function  $G$  may be assumed invariant under any permutation of its arguments, and we may take  $C_{\sigma\sigma'}(\mathbf{p}) = C_{\sigma'\sigma}(-\mathbf{p})$ . Moreover, up to a vacuum renormalization, we may assume  $G(0, \mathbf{p}, -\mathbf{p}) = 0$  in the trial state. An exact decoupling can be easily achieved by the method of Ref. [12], thus avoiding any further approximation. The full details will be published elsewhere. Here we discuss the results in the two special cases  $C = 0$  and  $G = 0$ .

For  $C = 0$  there is no decay and the trial state (1) is an improvement over the  $|hh\rangle + |hhh\rangle$  variational ansatz of Ref. [12]. Here we have one extra equation arising from the variation with respect to  $A$  in Eq.(6). However the extra coefficient  $A$  is a constant which can be easily eliminated yielding two coupled integral equations. We regularize the logarithmically divergent integrals with an energy cut-off  $\omega_p = \sqrt{\mathbf{p}^2 + m^2} < \Lambda$ . Neglecting terms of order  $\mathcal{O}(\Lambda^{-2})$ , the method of Ref. [12] allows an exact decoupling of the integral equations yielding

$$(2\omega_k - E)B(\mathbf{k}) = - \int d^3p \mathcal{K}(\mathbf{k}, \mathbf{p}, -\mathbf{k} - \mathbf{p})B(\mathbf{p}), \quad (8)$$

where the kernel  $\mathcal{K}$  is defined as

$$\begin{aligned} \mathcal{K}(\mathbf{k}, \mathbf{p}, \mathbf{q}) = & \frac{1}{64\pi^3\omega_k} \left( \frac{2m_0^2 + m^2}{v^2} \right) \times \\ & \times \left\{ \left( \frac{2m_0^2 - 2m^2}{2m_0^2 + m^2} \right) \frac{1}{\omega_p} - \frac{2}{\omega_p\omega_q} \frac{m^2 + 2m_0^2 + \omega_p(E - 2\omega_p)}{\left[ \omega_k + \omega_p + \omega_q + \frac{m_0^2 - m^2}{2} \left( \frac{1}{\omega_k} + \frac{1}{\omega_p} + \frac{1}{\omega_q} \right) - E \right]} \right\}. \end{aligned} \quad (9)$$

This differs from the  $|h\rangle + |hh\rangle$  calculation of Ref. [12] for a decrease of the numerical coefficient of the first (repulsive) term inside the brackets. In Eq.(8) a self-consistency condition has been imposed in order to fix the lower bound  $E_0$  of the continuous spectrum of two-particle scattering states. Imposing  $E_0 = 2m$  yields the mass renormalization condition

$$m^2 = m_0^2 \left[ \frac{1 - 2J(0)}{1 + J(0)} \right] \quad (10)$$

where

$$J(0) = \frac{\lambda}{32\pi^2} \int_1^{\Lambda/m} \frac{\sqrt{x^2 - 1}}{x^2 - \alpha x - \beta} dx, \quad (11)$$

$\alpha = (3 - m_0^2/m^2)/4$  and  $\beta = (1 - m_0^2/m^2)/2$ . These conditions ensure that the integral equation (8) always admits the free-wave solution  $E = 2m$  as the lower bound of the continuous spectrum. The numerical solution of the coupled equations (10), (11) is reported in Fig.1 for a large cut-off  $\Lambda = 14$  TeV. The perturbative approximation  $m \approx m_0$  breaks down for  $m_0 > 0.3$  TeV. Moreover, in the strong coupling limit, we find a light  $m < 100$  GeV for any  $m_0 > 1.9$  TeV. Thus a physical mass  $m \approx 100$  GeV could result from very small or very large couplings. The strong coupling case is characterized by the presence of bound state solutions, i.e. two-particle solutions of Eq.(8) with  $E < 2m$ . In Fig.2 the binding energy is reported and compared to the prediction of the  $|hh\rangle + |hhh\rangle$  ansatz. In the present calculation, the presence of the extra term  $|h\rangle$  represents an improving of the trial state, and causes a decrease of the binding energy as it should be expected for any variational calculation.

In order to study the decay width we must restore  $C \neq 0$  in the trial state (1). Here we prefer to discuss the  $G = 0$  case for brevity. For the  $|h\rangle + |hh\rangle + |ZZ\rangle$  state the eigenvalue equations can be easily decoupled yielding

$$C_{\sigma\sigma'}(\mathbf{p}) = [e^\mu(\mathbf{p}\sigma)e_\mu(-\mathbf{p}\sigma')]^* \left[ \delta^3(\mathbf{p} - \mathbf{p}_0) - \Delta(E)f(\mathbf{p}, E)\rho(E) \right] \quad (12)$$

$$\Delta(E) = \sum_{\sigma\sigma'} \int \frac{d^3p}{\Omega_p} e^\mu(\mathbf{p}\sigma)e_\mu(-\mathbf{p}\sigma') C_{\sigma\sigma'}(\mathbf{p}) \quad (13)$$

$$f(\mathbf{p}, E) = \left( \frac{M}{v} \right)^4 \frac{v^2}{32\pi^3 m(E - m)\Omega_p(2\Omega_p - E)} \quad (14)$$

$$\rho(E) = \frac{(2m_0^2 + m^2)^2}{9m_0^4} \left[ 1 - \frac{2m(E - m)(m_0^2 - m^2)}{(2m_0^2 + m^2)^2} \right] \quad (15)$$

where  $e_\mu(\mathbf{p}\sigma)$  are the polarization vectors,  $\Omega_p = \sqrt{\mathbf{p}^2 + M^2}$  and  $E = 2\Omega_{p_0}$ . The right hand side of Eq.(12) may be interpreted as the sum of a free wave and a scattered wave for the process  $Z^0 Z^0 \rightarrow h \rightarrow Z^0 Z^0$ . The scattered wave yields [16,8] the cross-section and the decay width of the Higgs boson which appears as a resonance for  $m > 2M$ . Eq.(12) is an integral equation since, according to Eq.(13),  $\Delta(E)$  is an integral functional of the wave function

$C$ . Even in the strong coupling limit, the small parameter  $M/v$  in Eq.(14) allows the usual perturbative expansion obtained by iteration. Thus, at leading order, substituting Eq.(12) in Eq.(13) gives

$$\Delta(E) = \frac{2}{E} \left( 3 - \frac{E^2}{M^2} + \frac{E^4}{4M^4} \right) + \mathcal{O}(M^4/v^4) \quad (16)$$

Let us explore this result in the two opposite limits of very weak ( $m_0 \approx m$ ) and very strong ( $m_0 \gg m$ ) self-coupling. For  $m = m_0$  the coefficient  $\rho(E) = 1$  and the scattered wave  $\Delta(E)f(\mathbf{p}, E)$  becomes identical to that obtained by Di Leo and Darewych [8]. The cross-section is highly resonant near  $E = m$  and can be fitted by the Breit-Wigner formula yielding [8] a decay width  $\Gamma_{BW}$  identical to that obtained from covariant perturbation theory [18–20]:

$$\Gamma_{BW} = \frac{m^3}{32\pi v^2} \left( 1 + \mathcal{O}(M^2/m^2) \right) \quad (17)$$

Thus in the perturbative limit the present variational calculation and standard covariant perturbation theory are in perfect agreement. In the opposite strong-coupling regime we already know that according to Fig.1 the physical Higgs mass  $m$  can be considerably less than the bare mass  $m_0$ . The self-coupling  $\lambda$  enter the scattered wave in Eq.(12) only through the bare mass  $m_0$  in the factor  $\rho(E)$ . Even in the very strong coupling limit,  $\rho(E)$  does not change too much and is of order unity. For  $m, E \ll m_0$  it takes the limit value  $\rho(E) \approx 4/9$ . The non-perturbative decay width follows [8] as  $\Gamma_{NP} = \rho(m)\Gamma_{BW} \approx (4/9)\Gamma_{BW}$ . Thus, apart from the prefactor  $\rho$ , the decay width is obtained by inserting the renormalized Higgs mass  $m$  in the standard perturbative result (17). As a consequence, whatever is the strength of the self-coupling  $\lambda$ , if the physical Higgs mass is small the decay width remains small in the  $Z^0 - Z^0$  resonance. We do not see how this scenario could be changed by the inclusion of other processes.

Our findings are not in disagreement with the so called *equivalence theorem* [21,18,22] which states that at high energies the scattering amplitudes of longitudinal bosons are equivalent to the scattering amplitudes of their corresponding would-be Goldstone bosons.

In fact, the Higgs boson and the Goldstone bosons are coupled by the same interaction strength  $\lambda$ , which is assumed to be large in the strong-coupling limit. However in the Higgs mechanism the Goldstone bosons are not physical since the corresponding degrees of freedom are taken by the longitudinal polarizations of the massive vector bosons. It is only at high energy that the scattering amplitudes of the longitudinal gauge bosons are well described by the unphysical amplitudes of the Goldstone bosons. In the strong-coupling limit the Higgs mass is kept small by the renormalization effect, and the Higgs resonance at  $E = m$  is a low energy process which cannot be described by use of the equivalence theorem.

We must mention that, by  $1/N$  expansion in the strong-coupling limit, Ghinculov and Binoth [11] find a large decay width that increases with  $\lambda$  even beyond the saturation of  $m$ . These authors do not explore the very strong coupling regime where the Higgs mass is small. Besides, their expansion starts from a  $O(N)$  symmetric sigma model which contains the unphysical Goldstone bosons, and their calculation contains a tachyonic pole which is regularized by a perturbative method. Thus it is not clear if their method can be regarded as a genuine non-perturbative approximation, and if their finding can be compared to our low energy calculation for the decay width.

The existence of a quite extended strong-coupling range, where the physical Higgs mass  $m$  is small, increases the chances of detecting the Higgs boson below the TeV scale. However a strongly interacting light Higgs would differ from a weakly coupled one for several detectable aspects. For instance the existence of bound states would be the signature of a strongly interacting Higgs sector. While perturbation theory would be enough for a weakly interacting boson, the role of non-perturbative calculations would be determinant if the Higgs field turns out to be strongly self-coupled.

In summary, by a non-perturbative variational method we have shown that in the strong-coupling limit the mass of the Higgs boson would be small as a consequence of mass renormalization. Moreover the decay process at  $E \approx m$  would be a low energy process characterized by a small decay width. Thus, in order to establish if the Higgs sector is weakly or strongly interacting, the eventual detection of a light Higgs boson will not be enough, and



the more general phenomenology has to be considered and compared with the predictions of non-perturbative calculations.

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## FIGURES

FIG. 1. The physical Higgs mass  $m$  versus the bare mass  $m_0$  (which fixes the coupling strength), for an energy cut-off  $\Lambda = 14$  TeV. The dotted line represents the tree-level perturbative approximation  $m = m_0$ , which only holds in the weak-coupling regime  $m_0 < 0.3$  TeV.

FIG. 2. Higgs-Higgs binding energy  $E - 2m$  in units of  $2m$  versus physical Higgs mass  $m$  for a cut-off  $\Lambda = 14$  TeV, according to Eq.(8) of the text (squares). For comparison, the binding energy obtained by the simpler  $|hh\rangle + |hhh\rangle$  trial state is reported (circles). Notice that the binding energy decreases as the physical mass increases, since this last one is a decreasing function of the coupling strength according to Fig.1



